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INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

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N° 5270

Juillet 2004

Thème NUM

Rapport de recherche

ISRN INRIA/RR--5270--FR+ENG

N 0249-6399

Theoretical and Numerical Analysis of a Class of Nonlinear Elliptic Equations

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Thème NUM — Systèmes numériques
Projet CALVI

Rapport de recherche n° 5270 — Juillet 2004 — 20 pages

Abstract: In this paper we show the existence of weak solutions for a nonlinear elliptic equations with arbitrary growth of the non linearity and data measure. A numerical algorithm to compute a numerical approximation of the weak solution is discribed and analysed. In a first step a super-solution is computed using a domain decomposition method. Numerical examples are presented and commented.

Key-words: Domain decomposition, Nonlinear elliptic PDE

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Analyse théorique et numérique d'une classe de équations non linéaires elliptiques

Résumé : Dans ce papier on montre l'existence de solutions faibles d'une équation elliptique non lin eaire avec une croissance arbitraire de la non linéairité et des données mesure. Un algorithme nous permettant de calculer une approximation d'une solution faible est présenté et analysé. Dans un premier pas de l'algorithme on calcule une supersolution à l'aide d'une méthode de décomposition de domaines. Par ailleurs on présente quelques exemples numériques.

Mots-clés : Equations Différentielles Non Linéaires, Décomposition de Domaines

1 Introduction

The principal objective of this work is to study existence, uniqueness and present a numerical analysis of weak solutions for the following quasilinear elliptic problem:

$$\left\{ \begin{array}{l} -u''(t) + G(t, u'(t)) = F(t, u(t)) + f \;\; in \;\; (0,1) \\ u(0) = u(1) = 0 \end{array} \right. \tag{1}$$

where $G, F : [0,1]\times R \to [0,+\infty[$ are measurable and continuous with respect to u' and u, f is a given finite non negative measure on $(0,1)$. Such problems arises from biological, chemical and physical systems and various methods have been proposed for study the existence, uniqueness, qualitative properties and numerical simulation of solutions(see [11], [14]). When f is regular, it is prouved in [12] that if (1) has a nonnegative supersolution in $W_0^{1,\infty}$ then (1) has a solution in $W_0^{1,\infty} \bigcap W^{2,p}$. Note that here the supersolution is required to vanish at the boundary. This provides an a priori pointwise estimate for $u'(0)$ and $u'(1)$. The boundedness on u' on the whole set $(0,1)$ is then obtained by a maximun principle applied to the equation satisfied by $|u'|^2$. The convexity of $s \to G(t,s)$ is the essential ingredient. Many authors dealt with this problem when f is irregular and G is subquadratic with respect to u' namely:

$$|G(t,r)| \leq c(g(t) + |r|^2), \quad g(t) \in L^1(0,1), c > 0 \tag{2}$$

They showed that, if G satisfy(2), (1) has a solution $u \in H_0^1(0,1)$ provided that (1) has a supersolution in $W^{1,\infty}(0,1)$ see [5], [4] and the references there in.

The case where the supersolution itself is irregular have been treated in [2], it is a solution in $H_0^1(0,1)$ then (1) has a solution in $H_0^1(0,1)$ provided that G satisfy (2).

In this work we are particularly interested in situations where f is irregular and where the growth of G with respect to u' and F with respect to u are arbitrary. Let us make some precitions on model problem like:

$$\left\{ \begin{array}{l} -u''(t) + |u'(t)|^q = |u(t)|^p + f \;\; \text{in} \;\; (0,1) \\ u(0) = u(1) = 0 \end{array} \right. \tag{3}$$

where $p, q \geq 1$ and $f \in M_B^+(0,1)$, the set of nonnegative finite measure on $(0,1)$. We show here that if the semilinear problem:

$$\left\{ \begin{array}{l} -w''(t) = |w(t)|^p + f \;\; \text{in} \;\; (0,1) \\ w(0) = w(1) = 0 \end{array} \right. \tag{4}$$

has a solution then (3) has a solution. Remark here any restriction for p and q is imposed. For an elegant study of (4) one can sen the work of Pierre and Baras [7]. If $w'(0) = +\infty$ or $w'(1) = -\infty$ then $w \notin W_0^{1,\infty}$ and obviously the classical approach fails to provide existence in (3) and new techniques have to be used. We describe some of them here.

Another approach studied here is the numerical approximation of the solution to the problem (1). The most important dificulties are in this approach the uniqueness and the blowup of the solution.

The general algorithm for numerical solution of this equations is one application of the Newton method to the discretaized version of problem (1):

$$\text{Find } U \in R^m \text{ such that } AU = H(U) \tag{5}$$

where A is a sparce matrix and $H : R^m \to R^m$ is a nonlinear operator.

The Newton algorithm is given by:

$$\begin{cases} \text{chosse } U^0 \text{ in a neighbourhood of the solution} \\ \text{and solve until convergence} \\ (A - H'(U^k)\, Id)\,(U^{k+1} - U^k) = -A\, U^k + H(U^k) \end{cases} \tag{6}$$

where $H'(U^k)$ is the Jacobian matrix of the operator H computed in U^k and Id is a matrix of identity in R^m. This method converges quadraticaly when it converges. Convergences depend in particular in the choise of U^0 and the existence and uniqueness of solutions of the linear system (6). In the case of problem (1) the matrix $A - H'(U^k)Id$ is ofthen singular. Consider the following example:

$$\begin{cases} -u''(t) = \alpha\, u(t) + \beta \;\; in \;\; (0,1) \\ u(0) = u(1) = 0 \end{cases} \tag{7}$$

where α and β belongs to R. It is easy to verify that (7) have an infinity of solutions when $\alpha = (2\pi p)^2$. For all $B \in R$, $\exists\ u_B$ solution of (7) where:

$$u_B(t) = \frac{\beta}{\alpha}\,(1 - cos(p\,\pi\,t)) + B\,sin(\,p\,\pi\,t\,) \tag{8}$$

If we consider a classical discretization of u'' by a finite differences schema and choose α an eigenvalues of the matrix A. The Newton schema is written as follows:

$$(A - \alpha\, Id)\,(U^{k+1} - U^k) = -A\, U^k + H(U^k) \tag{9}$$

Clearly the matrix $A - \alpha\, Id$ is singular and the system (9) have not necessary a solution or a infinite number of solutions if $-A\, U^k + H(U^k) \in Im(A - \alpha\, Id)$.

To overcame this difficulty we introduce a domain decomposition to compute an approximation of $\delta u^k = u^{k+1} - u^k$ by the resolution of a sequence of problems of type (1) in subset Ω_i of $(0,1)$, such that $\Omega = \bigcup_{i=1,K} \Omega_i$. The idea of the method cames from the following remark [17]:

Lemma 1.1 *Let* $0 \le a < b \le 1$, $a_i \in L^\infty(0,1)$, *for* $i = 1,2$. *If* $|b-a|$ *is small enough then the operator* $-\dfrac{d^2}{dt^2} - a_1(t)\dfrac{d}{dt} - a_2(t)Id$ *have an inverse in* (a,b).

We have organized this paper in the following maner. In section 2 we give the precise setting of the problem, we present a approximate equation for (1) and we prove that the existence of weak supersolutions implies the existence of weak solutions, without any restriction of the growth of G with respect to u', this result generalise the classical result of [12], [5] and [2].

In section 3 we present an approximation scheme for problem (1) based on the Schwartz overlapping domaine decomposition method, combined with finite element method.

2 Mathematical analysis of the problem

Throughout this paper we suppose

$$f \quad \text{is a nonnegative finite measure on } (0,1) \tag{10}$$

and $G, F : [0,1] \times R \to [0,+\infty)$ are such that G, F are measurable

$$\text{The functions } \ r \to G(t,r), F(t,r) \text{ are continuous a.e. } \ t \tag{11}$$

$$F(t,.) \text{ is nondecreasing and } G(t,.) \text{ is convex,} \tag{12}$$

$$\forall r \in R,\ G(.,r),\ F(.,r) \ \text{ are integrable on } \ (0,1) \tag{13}$$

$$G(t,0) = \min\{G(t,r), r \in R\} = 0 \ \text{ and } F(t,0) = 0. \tag{14}$$

Now we introduce the notion of weak solution, supersolution and subsolution used here.

Definition 2.1 *A function u is said to be a weak solution of (1) if*

$$\left\{ \begin{array}{l} u \in W^{1,\infty}_{loc}(0,1) \bigcap C_0[0,1] \\ -u''(t) + G(t,u'(t)) = F(t,u(t)) + f \ \ in \ \mathcal{D}'(0,1) \end{array} \right. \tag{15}$$

(replace in (15) = by $\geq$ for a weak supersolution and by $\leq$ for a weak subsolution)

Remark 2.2 In (15) $u \in W^{1,\infty}_{loc}(0,1)$, using (13) we have $G(t,u'(t))$ and $F(t,u(t)) \in L^1_{loc}(0,1)$. Hence every term in (15) makes sense.

This enables us to state the main result of this paper.

Theorem 2.3 *Assume that (10)-(14) and $f \in M^+_B(0,1)$ hold. Assume that there exists a weak solution $\overline{w}$ for the problem,*

$$\left\{ \begin{array}{c} \overline{w} \in W^{1,\infty}_{loc}(0,1) \cap C_0[0,1] \\ -\overline{w}'' = F(.,\overline{w}) + f \quad \ in \ \ \mathcal{D}'(0,1) \end{array} \right. \tag{16}$$

Then $\overline{w}$ is a supersolution of (1) and there exist a weak solution u of (1) such that $u \leq \overline{w}$.

Remark 2.4 1) It should be noted that there is not growth restriction on the lower order nonlinearity of F and G w.r.t. u and u' respectively. Hence the present theorem extends some results in [2], [5].

2)For any finite nonnegative measure f, the problem:

$$\left\{ \begin{array}{l} \underline{w} \in W^{1,\infty}_0(0,1), \quad \underline{w} \geq 0 \ \text{ in } \ (0,1) \\ -\underline{w}'' + G(t,\underline{w}') = f \quad \text{in} \ \ \mathcal{D}'(0,1) \end{array} \right. \tag{17}$$

has a unique solution $\underline{w}$, see [1], and remark here that $\underline{w}$ is a subsolution of the problem (15).

2.1 An approximate equation

For $n \geq 0$, we consider the Yosida approximation $G_n(t,.)$ of $G(t,.)$ defined by:

$$G_n(t,r) = \begin{cases} G(t,-n) + G_r'(t,-n)\,(r+n) & \text{if } r \leq -n \\ G(t,r) & \text{if } |r| < n \\ G(t,n) + G_r'(t,n)\,(r-n) & \text{if } r \geq n \end{cases} \tag{18}$$

where $G_r{}'$ denotes a section of the subdifferential of G with respect to r.

Then G_n satisfies (11) -(14) and

$$G_n \leq G \ , \ G_n \leq G_{n+1} \tag{19}$$

then $G_n(t,.)$ increases a.e. to $G(t,.)$ as n tends to infinity.

According to the result in [1], [5] there exists a sequence (u_n) of solution of the problem:

$$\begin{cases} u_{n+1} \in W_0^{1,\infty}(0,1) \\ -u_{n+1}'' + G_{n+1}(t,u_{n+1}') = F(u_n) + f \quad \text{in} \quad \mathcal{D}'(0,1) \end{cases} \tag{20}$$

where $u_0 = \overline{w}$.

2.2 Estimates-Passing to the limit

In order to proof the theorem 2.3 we propose to send n to infinity in (20). For this we will need some estimates passing to the limit.

Lemma 2.5 *Let $a(t) \in L^1_{loc}(0,1)$, $v \in W^{1,1}_{loc}(0,1) \cap C_0[0,1]$ such that*

$$\begin{cases} a(t)\,v'(t) \in L^1_{loc}(0,1) \\ -v'' - a\,v' \geq 0 \quad in \quad \mathcal{D}'(0,1) \end{cases} \tag{21}$$

Then $v \geq 0$ in $[0,1]$. See a proof in [1]

Lemma 2.6 *Let $u \in W^{1,1}_{loc}(0,1)$, $\underline{v}, \overline{v} \in L^\infty(0,1)$ and $\mu \in M_B^+(0,1)$ such that:*

$$\begin{cases} \underline{v} \leq u \leq \overline{v} \quad in \quad (0,1) \\ -u'' \leq \mu \quad in \quad \mathcal{D}'(0,1) \\ -\overline{v}'' \geq \mu \quad in \quad \mathcal{D}'(0,1) \end{cases} \tag{22}$$

Then $u \in W^{1,\infty}_{loc}(0,1)$, and

$$|\,u'(t)\,| \leq \frac{1}{d(t;a,b)}\,(c(a,b) + ||\underline{v}||_{L^\infty} + ||\overline{v}||_{L^\infty} + ||\mu||_{M_B}) \tag{23}$$

for all $0 < a < b < 1$. Where $d(t;a,b) = min(b-t, t-a)$ and $c(a,b)$ is a constant depending on a and b.

Lemma (2.6), will provide $W^{1,\infty}_{loc}(0,1)$ estimates for the approximate solution u_n. But this estimate don't allow us to pass to the limit in the nonlinear terms. We need the strong convergence of u_n in $W^{1,\infty}_{loc}(0,1)$. We obtain this result from the following Lemma.

Lemma 2.7 *Let* $(u_n)_n \subset W_0^{1,\infty}(0,1)$ *such that,*

$$u_n \to u \quad \textit{strongly in} \quad L^\infty(0,1) \tag{24}$$

$$\left\{\begin{array}{l} \underline{w} \le u \le u_n \le \overline{w} \\ -u'' \le \mu \quad \textit{in} \quad \mathcal{D}'(0,1) \\ -\overline{w}'' \ge \mu \quad \textit{in} \quad \mathcal{D}'(0,1) \end{array}\right. \tag{25}$$

Then $u'_n \to u'$ *strongly in* $L^\infty_{loc}(0,1)$

Proof of lemma (2.6). Let $0 < a < b < 1$ and let φ the capacity potential of $[a,b]$. The fonction $\theta = \overline{v} - u$ satisfies

$$\left\{\begin{array}{c} -\theta'' \ge 0 \text{ in } D'(0,1) \\ \theta \in W^{1,\infty}_{loc}(0,1) \cap L^\infty(0,1) \end{array}\right. \tag{26}$$

We have

$$\int_a^b -\theta'' = \int_a^b -\theta''\,\varphi \le \int_0^1 -\theta''\,\varphi = \int_a^b \theta\,\varphi'' \le c(a,b) + ||\overline{v}||_\infty + ||\underline{v}||_\infty \tag{27}$$

Then

$$\int_a^b -u'' = \int_a^b -\theta'' + \mu \le c(a,b) + ||\overline{v}||_\infty + ||\underline{v}||_\infty + ||\mu||_{M_B(0,1)} \tag{28}$$

Using a similar technique, we deduce that for $a < x < y < b$, we have

$$u'(x) - u'(y) \le c(a,b) + ||\overline{v}||_\infty + ||\underline{v}||_\infty + ||\mu||_{M_B(0,1)} \tag{29}$$

Integrate w.r.t. y over (x,b), to find:

$$\begin{array}{rcl} (b-x)u'(x) & \le & (b-x)\,(\,c(a,b) + ||\overline{v}||_\infty + ||\underline{v}\,||_\infty + ||\mu||_{M_B(0,1)}) \\ & \le & c(a,b) + ||\overline{v}||_\infty + ||\underline{v}||_\infty + ||\mu||_{M_B(0,1)} \end{array} \tag{30}$$

Integrate w.r.t. x over (a,y), to obtain

$$u(y) - u(a) \le (y-a)(c(a,b) + ||\overline{v}||_\infty + ||\underline{v}||_\infty + ||\mu||_{M_B(0,1)}) + (y-a)u'(y). \tag{31}$$

Then we deduce the following uniform local estimate

$$\forall x \in [a,b]\,, \quad ||\,u'(x)\,|| \le \frac{1}{d(x;a,b)}\,(c(a,b) + ||\overline{v}||_\infty + ||\underline{v}||_\infty + ||\mu||_{M_B(0,1)}) \tag{32}$$

where $d(x;a,b) = \min(x-a, b-x)$.

Proof of lemma (2.7). By lemma (2.6), we have $u \in W^{1,\infty}_{loc}(0,1)$ and

$$\forall x \in [a,b], \qquad ||\,u'(x)\,|| \le c(a,b) + ||\overline{v}||_\infty + ||\underline{v}||_\infty + ||\mu||_{M_B(0,1)}. \tag{33}$$

We then consider the function $\theta_n = \overline{w} - u_n$ satisfies the equation

$$\begin{cases} -\theta_n'' \geq 0 \text{ in } \mathcal{D}'(a,b) \\ 0 \leq \theta_n \leq \overline{w} - u \in [0,1] \end{cases} \tag{34}$$

Let φ the capacity potential of $[a,b]$, then we have:

$$\int_a^b |-\theta_n''| = \int_a^b -\theta_n'' = \int_a^b \theta'' \varphi \leq \int_0^1 -\theta'' \varphi \leq \int_0^1 -\theta_n \varphi'' \leq c(a,b) \tag{35}$$

$$\theta_n = \overline{w} - u_n \text{ converge to } \overline{w} - u \text{ in } L^\infty(0,1) \tag{36}$$

and applying Ascoli's theorem, the lemma follows.

Proof of the theorem (2.3). First we prove

$$\underline{w} \leq u_{n+1} \quad \text{for all} \quad n \geq 0 \tag{37}$$

Thanks to (20) and the definition of $\underline{w}$, we obtain

$$-(u_{n+1} - \underline{w})'' + G(u_{n+1}') - G(\underline{w}') \geq 0 \quad \text{in} \quad \mathcal{D}'(0,1) \tag{38}$$

using (11) we then have

$$\begin{cases} -(u_{n+1} - \underline{w})'' + a_n (u_{n+1} - \underline{w})' \geq 0 \quad \text{in} \quad \mathcal{D}'(0,1) \\ u_{n+1} - \underline{w} \in W_0^{1,1}(0,1) \\ a_n (u_{n+1}' - w') \in L^1(0,1) \end{cases} \tag{39}$$

where $a_n \in \partial G(., u_{n+1}') \in L^1(0,1)$. Now we can apply lemma (2.5), therefore $\underline{w} \leq u_{n+1}$ in $[0,1]$ wich proves (37).

Let us now prove by induction that

$$u_{n+1} \leq u_n \leq \overline{w} \quad \text{in} \quad [0,1] \quad \text{for all} \quad n \geq 0 \tag{40}$$

For $n = 0$, using (16), (20) we get

$$\begin{cases} \overline{w} - u_1 \in W_{loc}^{1,1}(0,1) \cap C_0[0,1] \\ -(\overline{w} - u_1) \geq 0 \quad \text{in} \quad \mathcal{D}'(0,1) \end{cases} \tag{41}$$

Applying lemma (2.5) we have $\overline{w} - u_1 \geq 0$. Let us assume $u_n \leq u_{n-1} \leq \overline{w}$, then from (20) and the monotonicity of F in r, we have

$$\begin{cases} u_n - u_{n+1} \in W_0^{1,1}(0,1); \ (\overline{w} - u_n) \in W_{loc}^{1,1}(0,1) \cap C_0[0,1] \\ -(u_n - u_{n+1})'' + G(u_{n+1}') - G(u_n') \geq 0 \quad \text{in} \quad \mathcal{D}'(0,1) \\ -(\overline{w} - u_n)'' - G(u_n') \geq 0 \quad \text{in} \quad \mathcal{D}'(0,1) \end{cases} \tag{42}$$

using now (12), (14) then we have from (42)

$$\left\{\begin{array}{l} u_n - u_{n+1} \in W_0^{1,1}(0,1) \\ -(u_n - u_{n+1})'' + a_n(u_{n+1} - u_n)' \geq 0 \quad \text{in} \quad \mathcal{D}'(0,1) \\ a_n \in \partial G(t, u'_{n+1}) \in L^1(0,1) \end{array}\right. \tag{43}$$

$$\left\{\begin{array}{l} \overline{w} - u_n \in W_{loc}^{1,1}(0,1) \cap C_0[0,1] \\ -(\overline{w} - u_n)'' \geq 0 \quad \text{in} \quad \mathcal{D}'(0,1) \end{array}\right. \tag{44}$$

Applying lemma (2.5), we deduce $u_{n+1} \leq u_n \leq \overline{w}$ in $[0,1]$ which proves (40) by induction. Employing lemma (2.6), we conclude that u_n is bounded in $W_{loc}^{1,\infty}(0,1) \cap C_0[0,1]$ independently of n. Therefore, there exists a subsequence, still denoted by (u_n) for simplicity, such that u_n converges to u strongly in $L^\infty(0,1)$ if $n \to \infty$. Also u'_{n+1} converges to u' strongly in $L^1_{loc}(0,1)$ and a.e. in $(0,1)$. Then from lemma (2.6) we conclude that u'_{n+1} converges to u' strongly in $L^\infty_{loc}(0,1)$, and

$$||u'_n||_{L^\infty(a,b)} \leq K(a,b)\,(\,c(a,b) + ||\overline{w}||_{L^\infty(0,1)} + ||f||_{M_B} + ||\underline{w}||_{L^\infty(0,1)}) \tag{45}$$

where $K(a,b) = 1/\eta$ and $0 < \eta < a < \eta + b < 1$.

Since $G(t,.)$ and $F(t,.)$ are continuous with respect the two last arguments, we have for all $0 < a < b < 1$

$$G(t, u'_{n+1})\,,\, F(t, u_n) \;\rightarrow\; G(t, u')\,,\, F(t,u) \text{ a.e. } t \in (0,1). \tag{46}$$

On the other hand, for a.e $t \in (a,b)$

$$|G(t, u'_{n+1}(t)| \leq \max_{|r| \leq C'(a,b)} |G(t,r)| = \theta(t) \tag{47}$$

and

$$|F(t, u_n(t))| \leq \max_{|s| \leq \max(\,||\overline{w}||_{L^\infty(0,1)},\, ||\underline{w}||_{L^\infty(0,1)})} |F(t,s)\,| = \widehat{\theta}(t) \tag{48}$$

and $\theta, \widehat{\theta} \in L^1_{loc}(0,1)$ from (13). Using Lebesgue's dominate convergence Theorem (see [6]), we also have;

$$G(t, u'_{n+1}),\; F(t, u_n) \;\rightarrow G(t, u'),\; F(t,u) \quad \text{in} \;\; L^1(a,b) \text{ respectively} \tag{49}$$

Now, we can pass to the limit in (20), and if $\varphi \in \mathcal{D}(0,1)$ with support of $\varphi \subset [a,b]$ then

$$\begin{array}{rcl} 0 & = & \lim_{n \to \infty} \langle -u''_{n+1} + G(u'_{n+1}) - F(u_n)\,,\, \varphi \rangle \\ & = & \langle\, -u'' + G(u') - F(u)\,, \varphi \,\rangle \end{array} \tag{50}$$

where $\langle .,. \rangle$ denotes the duality pairing between $\mathcal{D}'(0,1)$ and $\mathcal{D}(0,1)$. This completes the proof.

3 Numerical method

3.1 Introduction

In this section we present the numerical method to solve the equation (1). Formally the iterative method construct a sequence of numerical solutions of (20) in $H_0^1(0,1)$ with a first guess wich is a supersolution of (1), in our case a solution of the problem (16).

Then the algorithm can be formulated in the following way:

1) Find $\overline{w} \in H_0^1(0,1)$ such that:

$$-\,\overline{w}''(t) \;\geq\; F(t,\overline{w}) \,+\, f \;\text{ in } (0,1) \tag{51}$$

2)Given $u_0 \;=\; \overline{w}$ we compute a sequence, (u_n), solution in $H_0^1(0,1)$ of the non linear equation:

$$-\; u''_{n+1}(t) \,+\, G_{n+1}(t,u'_{n+1}) \;=\; F(t,u_n) \,+\, f \;\text{ in } (0,1) \tag{52}$$

Both problems (51) and (52) are nonlinear, and if (51) have a solution, in theorem 2.3 we prove that (52) have also a solution. Let us start by considering the numerical resolution of problem (51).

3.2 Numerical resolution of equation (51)

To solve the nonlinear equation (51), which presents some interesting difficulties, we consider the Newton method. We construct a sequence $\overline{w}^k$ such that $\overline{w}^k$ is a solution of a linear problem and $\overline{w}^k$ converges to $\overline{w}$.

Let $\overline{w}^0 \;=\; 0$, we define $\overline{w}^{k+1} \;=\; \overline{w}^k \,+\, \delta$ where δ is the solution of the following linear problem:

$$\left\{\begin{array}{l} -\delta''(t) - \frac{\partial F(t,\overline{w}^k)}{\partial r}\delta(t) = (\overline{w}^k)''(t) + F(t,\overline{w}^k) + f \quad \text{in} \quad (0,1) \\ \delta(0) = \delta(1) = 0 \end{array}\right. \tag{53}$$

Then each iteration we have to solve the linear problem (53). To this aim we considered a weak formulation of the problem and finite element method.

To simplify the text we reformulate (53) in the following way: find $v \in H_0^1(a,b)$ such that:

$$\left\{\begin{array}{rl} -\,v(t)'' \,+\, c(t)\,v(t) \;=\; h & \text{in } (a,b) \\ v(a) = v(b) \;=\; 0 & \end{array}\right. \tag{54}$$

where $h \in M_B(a,b)$, the set of finite mesure in (a,b), and $c(t) \in L^2(a,b)$, without any restriction in it sign. We assume $c_\infty = ||c||_{L^\infty(a,b)}$ bounded.

In the previous section, Lemma 1.1, saids that the problem (54) have a solution in a domain (a,b) small enough.

If $V \;=\; H_0^1(a,b)$ then the weak formulation (54) reads:

$$\text{find } \; v \in V \;:\; a(v,w) = (h,w) \;\; \forall\, w \,\in\, V \tag{55}$$

where:

$$(v,w) = \int_a^b u\,v\,dx \tag{56}$$

$$a(v,w) = (v',w') + (c(t)\,v,w) \tag{57}$$

Thanks to the Poincaré inequality we have:

$$(v',w') = \|w'\|^2_{L^2(a,b)} \geq \frac{c_o}{|b-a|}\,\|w\|^2_{L^2(a,b)} = \frac{c_o}{|b-a|}(v,w) \tag{58}$$

and in the case of the bilinear form $a(w,v)$ we obtain:

$$a(w,w) = (w',w') + (c\,w,w) \geq (\frac{c_o}{|b-a|} - c_\infty)(w,w) \tag{59}$$

Then the bilinear form $a(w,v)$ should be coercive if $|b-a| < \dfrac{c_0}{c_\infty}$.

This remark are of great interest, because they can be exploited to obtain a numerical solution of (54) using a domain decomposition technique. In other words, this means that the domain partition should be determined by the behavior of $\left\|\frac{\partial F(\overline{w}^k)}{\partial r}\right\|_\infty$.

The aim of this section is to introduce the Schwarz overlapping domain decomposition method [15] applied to problem (54).

First we decompose (a,b)in a set of m overlapping sub dommains $(a_i\,,\,b_i)$such that $(a,b) = \cup_{i=1}^m (a_i\,,\,b_i)$and $(a_i\,,\,b_i) \cap (a_{i+1}\,,\,b_{i+1}) \neq \emptyset$

Then, if $v^0$is an initialisation function defined in $(a,b)$and vanishing in $a$and $b$we define for $k \geq 0$, m sequences v_i^k, $i = 1,...m$ solving the following problems:

$$\begin{cases} -(v_1^{k+1})''(t) + c(t)\,v_1^{k+1}(t) = h \text{ in } (0,b_1) \\ v_1^{k+1}(a) = 0;\; v_1^{k+1}(b_1) = v_2^k(b_1) \end{cases} \tag{60}$$

for $i = 2,...m-1$

$$\begin{cases} -(v_i^{k+1})''(t) + c(t)\,v_i^{k+1}(t) = h \text{ in } (a_i,b_i) \\ v_i^{k+1}(a_i) = v_{i-1}^{k+1}(a_i);\; v_i^{k+1}(b_i) = v_{i+1}^k(b_i) \end{cases} \tag{61}$$

and

$$\begin{cases} -(v_m^{k+1})''(t) + c(t)\,v_m^{k+1}(t) = h \text{ in } (a_m,1) \\ v_m^{k+1}(a_m) = v_{n-1}^{k+1}(a_m);\; v_m^{k+1}(b) = 0 \end{cases} \tag{62}$$

The variationnal formulation of the overlapping Schwarz method for the problem (53) can be stated as follows, set $V = H_0^1(a,b)$, $V_i^0 = H_0^1(a_i,b_i)$, $i = 1,...m$and

$$a_i(v,w) = \int_{a_i}^{b_i} v'(t)\,w'(t) + c(t)\,v(t)\,w(t)\,dt \tag{63}$$

Given $v^0 \in V$, solve for each $k \geq 0$:

$$\eta_1^k \in V_1^0 : a_1(\eta_1^k, w_1) = (f, w_1) - a_1(v^k, w_1);\ \forall w_1 \in V_1^0 \tag{64}$$
$$v^{k+\frac{1}{2}} = v^k + \tilde{\eta}_1^k \tag{65}$$

for $i = 2, ...m-1$

$$\eta_i^k \in V_i^0 : a_i(\eta_i^k, w_i) = (f, w_i) - a_i(v^k, w_i);\ \forall w_i \in V_i^0 \tag{66}$$
$$v^{k+\frac{1}{2}} = v^k + \tilde{\eta}_i^k \tag{67}$$

$$\eta_m^k \in V_m^0 : a_i(\eta_m^k, w_m) = (f, w_m) - a_i(v^k, w_m);\ \forall w_m \in V_m^0 \tag{68}$$
$$v^{k+1} = v^k + \tilde{\eta}_m^k \tag{69}$$

where $\tilde{\eta}_i^k$denotes the extension of η_i^k by 0 in $(a, b) \setminus (a_i, b_i)$.

To simplify, without lost of generality, we assume that we can consider a two domains decomposition $(a, b) = (a, \beta) \bigcup (\alpha, b)$ such that:

$$\alpha < \beta \text{ and } (\beta - a), (b - \alpha) < min(\frac{c_0}{c_\infty}, \frac{\pi}{2\sqrt{c_\infty}}) \tag{70}$$

Then, if $v^0$is an initialization function defined in (a, b) and vanishing in a and b we define for $k \geq 0$, 2 sequences v_i^k, $i = 1, 2$ solving the following problems:

$$\left\{ \begin{array}{c} -(v_1^{k+1})''(t) + c(t)\, v_1^{k+1}(t) = h \text{ in } (a, \beta) \\ v_1^{k+1}(a) = 0;\ v_1^{k+1}(\beta) = v_2^k(\beta) \end{array} \right. \tag{71}$$

and

$$\left\{ \begin{array}{c} -(v_2^{k+1})''(t) + c(t)\, v_2^{k+1}(t) = h \text{ in} (\alpha, b) \\ v_2^{k+1}(\alpha) = v_1^k(\alpha);\ v_2^{k+1}(b) = 0 \end{array} \right. \tag{72}$$

Now to prove the convergence of the Schwarz overlapping domain decomposition algorithm applied to problem (54) we consider two problems:

$$\left\{ \begin{array}{l} -v_1(t)'' + c(t)\, v_1(t) = h \in (a, \beta) \\ v_1(a) = 0 ;\ v_1(\beta) = v_2(\beta) \end{array} \right. \tag{73}$$

and:

$$\left\{ \begin{array}{l} -v_2(t)'' + c(t)\, v_2(t) = h \text{ in } (\alpha, 1) \\ v_2(\alpha) = v_1(\alpha),\ v_2(b) = 0 \end{array} \right. \tag{74}$$

Let v be

$$v = \left\{ \begin{array}{l} v_1 \text{ in } (a, \beta) \\ v_2 \text{ in } (\alpha, b) \end{array} \right. v_1 = v_2 \text{ in } (\alpha, \beta) \tag{75}$$

With the restriction (70) we can suppose the existence of a solution of (73) in $C(a, \beta)$and a solution of (74) in $C(\alpha, b)$.

Theorem 3.3 *Assume a, b, α and β with the restriction (70). Then the sequence v^k converges to v in $C(a, \beta)$ and $C(\alpha, b)$.*

Proof:

Let $d^k = v_1^k - v$ in (a, β) and $e^k = v_2^k - v$ in (α, b).

We prove the following inequality:

$$||d^{k+2}||_\infty \leq \gamma\, ||d^k||_\infty \;\text{ and }\; ||e^{k+2}||_\infty \leq \gamma\, ||e^k||_\infty \tag{76}$$

where $\gamma < 1$.

The difference d^k satisfies the following equation.

$$\left\{ \begin{array}{c} -\, d^{k+1}(t)'' + c(t)\, d^{k+1}(t) = 0 \;\text{ in }\; (a, \beta) \\ d^{k+1}(a) = 0 \;\text{ and }\; d^{k+1}(\beta) = v_2^k(\beta) - v(\beta) = e^k(\beta) \end{array} \right. \tag{77}$$

and e^k satisfies a similar equation in (α, b):

$$\left\{ \begin{array}{c} -\, e^{k+1}(t)'' + c(t)\, e^{k+1}(t) = 0 \;\text{ in }\; (a, \beta) \\ e^{k+1}(\alpha) = v_1^k(\alpha) - v(\alpha) = d^k(\alpha) \;\text{ and }\; e^{k+1}(b) = 0 \end{array} \right. \tag{78}$$

If we consider the following equation:

$$\left\{ \begin{array}{l} -\, \varphi(t)'' - c_\infty\, \varphi(t) = 0 \;\text{ in }\; (a, \beta) \\ \varphi(a) = 0 \;\text{ and }\; \varphi(\beta) = |e^{k+1}(\beta)| \end{array} \right. \tag{79}$$

then $\varphi(t) = |e^{k+1}(\beta)| \dfrac{sin(\sqrt{c_\infty}\,(t-a))}{sin(\sqrt{c_\infty}\,(\beta-a))}$, this solution is unique and positive if $(\beta - a) < \dfrac{\pi}{2\sqrt{c_\infty}}$. In that case $||\varphi||_\infty = |e^{k+1}(\beta)|$.

The difference $z = \varphi - d^{k+2}$ is the solution of:

$$\left\{ \begin{array}{c} -\, z(t)'' + c(t)\, z(t) = (c(t) + c_\infty)\, \varphi(t) \;\text{ in }\; (a, \beta) \\ z(a) = 0 \;\text{ and }\; z(\beta) = |e^{k+1}(\beta)| - e^{k+1}(\beta) \end{array} \right. \tag{80}$$

Clearly $z \geq 0$ if $(\beta - a), (b - \alpha) < min(\dfrac{c_0}{c_\infty}, \dfrac{\pi}{2\sqrt{c_\infty}})$. Then $d^{k+2} \leq \varphi \leq |e^{k+1}(\beta)|$.

If now $z = \varphi + d^{k+2}$ we have

$$\left\{ \begin{array}{c} -\, z(t)'' + c(t)\, z(t) = (c(t) + c_\infty)\, \varphi(t) \;\text{ in }\; (a, \beta) \\ z(a) = 0 \;\text{ and }\; z(\beta) = |e^{k+1}(\beta)| + e^{k+1}(\beta) \end{array} \right. \tag{81}$$

Also $z \geq 0$ and $-\varphi \leq d^{k+2}(t)\,,\ \forall\, t \in (a, \beta)$.

Then the inequality $||d^{k+2}||_\infty \leq |e^{k+1}(\beta)| \leq ||e^{k+1}||_\infty$ holds.

To prove that $|e^{k+1}(\beta)| \leq \gamma\, ||d^k||_\infty$ with $\gamma < 1$ we consider the equation:

$$\left\{ \begin{array}{c} -\, \phi(t)'' - c_\infty\, \phi(t) = 0 \;\text{ in }\; (\alpha, b) \\ \phi(\alpha) = |d^k(\alpha)| \;\text{ and }\; \phi(b) = 0 \end{array} \right. \tag{82}$$

The solution of this equation is given by: $\phi(t) = |d^k(\alpha)| \dfrac{sin(\sqrt{c_\infty}\,(b-t))}{sin(\sqrt{c_\infty}\,(b-\alpha))}$. This solution is positive if $(b-\alpha) < \dfrac{\pi}{2\sqrt{c_\infty}}$ and then $\phi(t) \geq |e^{k+1}(t)|\ \forall\ t \in (\alpha, b)$.

At this step we consider $z = \phi - e^{k+1}$. Then z is the solution of:

$$\begin{cases} -z(t)'' + c(t)\,z(t) = (c(t) + c_\infty)\,\phi(t) \quad \text{in } (\alpha, b) \\ z(\alpha) = |d^k(\alpha)| - d^k(\alpha)\ , \text{ and } \quad z(b) = 0 \end{cases} \tag{83}$$

Clearly $z \geq 0$ and then $\phi(t) \geq e^{k+1}(t)$ for all t in (α, b).

If now we consider $z = \phi + e^{k+1}$ we have also $z \geq 0$ because $z(t)$ is the solution of the following equation:

$$\begin{cases} -z(t)'' + c(t)\,z(t) = (c(t) + c_\infty)\,\phi(t) \quad \text{in } (\alpha, b) \\ z(\alpha) = |d^k(\alpha)| + d^k(\alpha)\ , \text{ and } \quad z(b) = 0 \end{cases} \tag{84}$$

Then $|e^{k+1}(t)| \leq \phi(t)$ in (α, b) and $|e^{k+1}(\beta)| \leq \phi(\beta) \leq \gamma\,|d^k(\alpha)|$ with $\gamma = \dfrac{sin(\sqrt{c_\infty}\,(b-\beta))}{sin(\sqrt{c_\infty}\,(b-\alpha)}$. The coefficient γ is smaller than one only if $\alpha < \beta$.

In conclusion with the restriction (70) we have $||d^{k+2}||_\infty < ||d^k||_\infty$.

Using the same technique we prove that $||e^{k+2}||_\infty < ||e^k||_\infty$ if we have (70). First we prove that $||e^{k+2}||_\infty \leq |d^{k+1}(\alpha)|$. To this aim we consider the equation:

$$\begin{cases} -\lambda(t)'' - c_\infty\,\lambda(t) = 0 \quad \text{in } (\alpha, b) \\ \lambda(\alpha) = |d^{k+1}(\alpha)| \quad \text{and} \quad \lambda(b) = 0 \end{cases} \tag{85}$$

The solution is given by $\lambda(t) = |d^{k+1}(\alpha)| \dfrac{sin(\sqrt{c_\infty}\,(b-t))}{sin(\sqrt{c_\infty}\,(b-\alpha))}$. This solution is positive if $b - \alpha < min(\dfrac{c_0}{c_\infty}, \dfrac{\pi}{2\sqrt{c_\infty}})$.

If $z(t) = \lambda(t) + e^{k+2}(t)$ then $z(t)$ is the solution of the following equation:

$$\begin{cases} -z(t)'' + c(t)\,z(t) = (c(t) + c_\infty)\,\lambda(t) \quad \text{in } (\alpha, b) \\ z(\alpha) = |d^{k+1}(\alpha)| + d^{k+1}(\alpha) \quad \text{and} \quad z(b) = 0 \end{cases} \tag{86}$$

Clearly $z \geq 0$ if $(b-\alpha) < min(\dfrac{c_0}{c_\infty}, \dfrac{\pi}{2\sqrt{c_\infty}})$.

If now $z(t) = \lambda(t) - e^{k+2}(t)$ then $z(t)$ is the solution of the following equation:

$$\begin{cases} -z(t)'' + c(t)\,z(t) = (c(t) + c_\infty)\,\lambda(t) \quad \text{in } (\alpha, b) \\ z(\alpha) = |d^{k+1}(\alpha)| - d^{k+1}(\alpha) \quad \text{and} \quad z(b) = 0 \end{cases} \tag{87}$$

then $z \geq 0$ if $(b-\alpha) < min(\dfrac{c_0}{c_\infty}, \dfrac{\pi}{2\sqrt{c_\infty}})$.

It is an easy consequence that $|e^{k+2}(t)| \leq \lambda(t)$ and we conclude $||e^{k+2}||_\infty \leq |d^{k+1}(\alpha)|$.

Now we prove that $|d^{k+1}(\alpha)| \leq \gamma\, |e^k(\beta)|$. To this aim we consider the following problem:

$$\left\{ \begin{array}{l} -\,\eta(t)'' - c_\infty\, \eta(t) = 0 \quad \text{in } (a,\beta) \\ \eta(a) = 0 \ \text{ and } \ \eta(\beta) = |e^k(\beta)| \end{array} \right. \tag{88}$$

The solution is given by $\eta(t) = |e^k(\beta)| \dfrac{sin(\sqrt{c_\infty}\,(\beta - t))}{sin(\sqrt{c_\infty}\,(\beta - a))}$. This solution is positive if $\beta - a < min(\dfrac{c_0}{c_\infty}, \dfrac{\pi}{2\sqrt{c_\infty}})$.

If $z(t) = \eta(t) + d^{k+1}(t)$ then it's the solution of the following equation:

$$\left\{ \begin{array}{l} -\,z(t)'' + c(t)\, z(t) = (c(t) + c_\infty)\, \eta(t) \quad \text{in } (a,\beta) \\ z(a) = 0 \ \text{ and } \ z(\beta) = |e^k(\beta)| + e^k(\beta) \end{array} \right. \tag{89}$$

Then $z \geq 0$ if $(\beta - a) < min(\dfrac{c_0}{c_\infty}, \dfrac{\pi}{2\sqrt{c_\infty}})$. In the same way if $z(t) = \eta(t) - d^{k+1}(t)$ it' the solution of:

$$\left\{ \begin{array}{l} -\,z(t)'' + c(t)\, z(t) = (c(t) + c_\infty)\, \eta(t) \quad \text{in } (a,\beta) \\ z(a) = 0 \ \text{ and } \ z(\beta) = |e^k(\beta)| - e^k(\beta) \end{array} \right. \tag{90}$$

Then $z \geq 0$ if $(\beta - a) < min(\dfrac{c_0}{c_\infty}, \dfrac{\pi}{2\sqrt{c_\infty}})$.

We obtain that $|d^{k+1}(t)| \leq \eta(t)$ for all $t \in (a,\beta)$ and then $|d^{k+1}(\beta)| \leq \gamma\, |e^k(\beta)|$ with $\gamma = \dfrac{sin(\sqrt{c_\infty}\,(\beta - \alpha))}{sin(\sqrt{c_\infty}\,(\beta - a)}$. The coefficient γ is smaller than one only if $\alpha < \beta$.

We conclude that the Schwarz overlapping domain decomposition method applied to the problem (54) converges.

3.3 Numerical solution of the problem (52)

Let V be $H_0^1(0,1)$ and $\bar{w} \in H_0^1(0,1)$ a solution of the problem (51). Appling the Newton method to solve the equation (52) we obtain the following algorithm:

$$\left\{ \begin{array}{l} u_0 = \bar{w} \\ u_{k+1}^0 = u_k \end{array} \right. \tag{91}$$

and u_{k+1} is the limit of the sequence:

$$u_{k+1}^{i+1} = u_{k+1}^i + \theta \tag{92}$$

where θ is the solution of the linear equation:

$$\left\{ \begin{array}{rcl} -\theta(t)'' + \frac{\partial G_{k+1}(t, {u_{k+1}^i}')}{\partial r}\theta(t)' & = & -\,G_{n+1}(t, {u_{k+1}^i}') + {u_{k+1}^i}'' \\ & + & F(t, u_k) + f \\ \theta(0) = \theta(1) = 0 & & \end{array} \right. \tag{93}$$

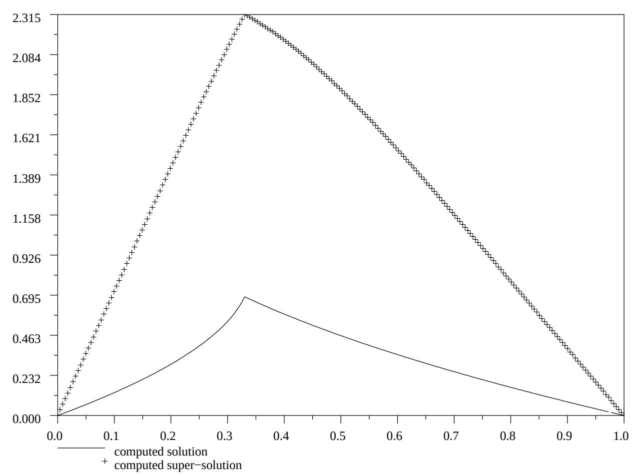

Figure 1: example $f = 8. * \delta_{\frac{1}{3}}$,m=10

3.4 Numerical Results

The algorithm introduced in the previous section has been implemented numerically for the model problem (3) with $p = q = 3$ and $f = \delta_{\frac{1}{3}}$.

$$\begin{cases} -u''(t) + |u'(t)|^q = |u(t)|^p + \delta_{\frac{1}{3}} \text{ in } (0,1) \\ u(0) = u(1) = 0 \text{ and } p = q = 3 \end{cases} \tag{94}$$

The number of subdomains is not fixed, its changes at each iteration according to the criterion (70). In figure 1 it can be observed the shape of the super-solution and the solution when the algorithm converges with $m = 10$ sub-domains.

To study the convergence history of the numerical simulation plotted in figure 1 we consider two steps. In the first step, where we compute a super-solution, we observe the evolution of the number of sub-domains: it goes from $m = 2$ sub-domains to $m = 10$ sub-domains in five iterations according to criterion (70). Simulation stops after 17 iterations when the residual is of the order 10^{-11}.

In the second step, starting with the super-solution computed in the previous step we perform nine iterations of the Yoshida approximation described in section 2 and the simulation stops when the correction computed is in uniform norm of the order 10^{-11}.

In figure 2 we consider the same exemple, but in that case we autotise a maximun of two subdomains. Clearly tha classical method fails to compute a solution.

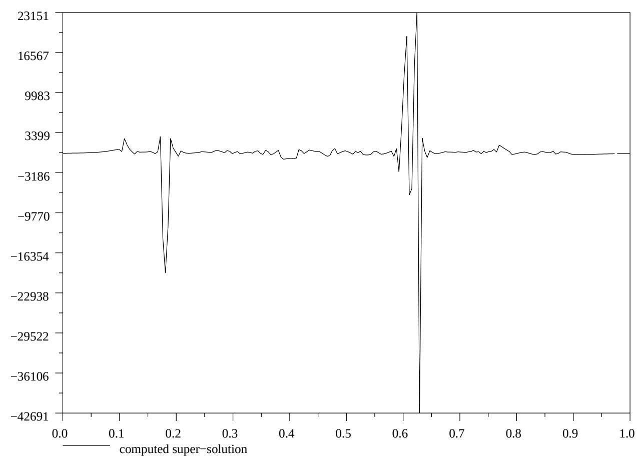

Figure 2: example $f = 8. * \delta_{\frac{1}{3}}$,m=2

In the next exemple we modifie the function G and F in the following way:

$$\left\{ \begin{array}{l} -u''(t) + \alpha(t)\,|u'(t)|^q = \beta(t)\,|u(t)|^p + f \quad \text{in} \quad (0,1) \\ u(0) = u(1) = 0 \end{array} \right. \tag{95}$$

where $p = 3,\ q = 4$ and:

$$\alpha(t) = \left\{ \begin{array}{c} 0 \ \text{ in } \ (0, 0.5) \\ 10 * (t - 0.5) \ \text{ in } \ (0.5, 1) \end{array} \right. \tag{96}$$

$$\beta(t) = \left\{ \begin{array}{c} 36 * (0.5 - t) \ \text{ in } \ (0, 0.5) \\ 0 \ \text{ in } \ (0.5, 1) \end{array} \right. \tag{97}$$

In figure 3 it can be observed the shape of the super-solution and the solution when the algorithm converges with $m = 7$ sub-domains. In the first step, where we compute a supersolution, we can observe in figure 4 the evolution of the number of subdomains required to satisfies criterion (70). At the first iteration the number of sub-domains are two, at the fifth iteration we have four sub-domains and afther the eighteenth iteration we reach seven sub-domains, the algorithme converges at the twentieth iteration.

Starting with the supersolution computed in the first step we perform eleven iteration of the second step (20). For each iteration of the Yosida approximation (20) we perform about three step of the Newton method. The simulation stops when the correction computed is in uniform norm of order 10^{-11}.

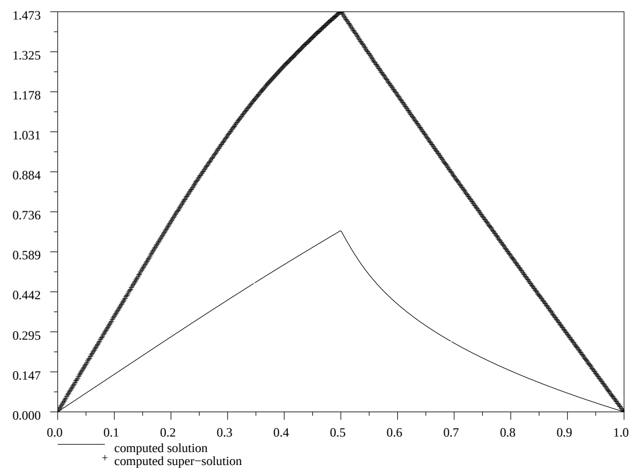

Figure 3: example $f = 5. * \delta_{\frac{1}{2}}$,m=7,p=3,q=4

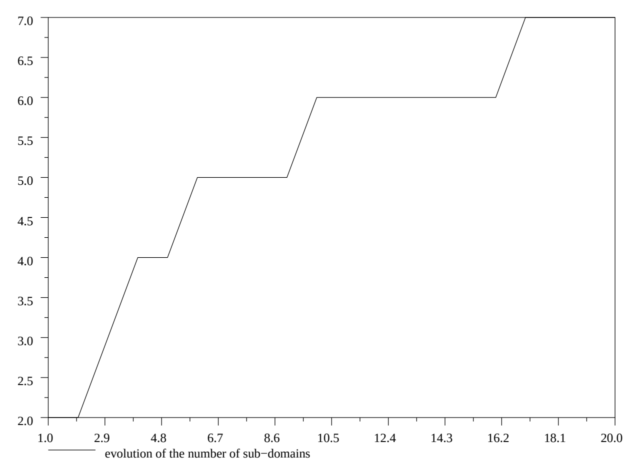

Figure 4: example $f = 5. * \delta_{\frac{1}{2}}$,m=7

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Acknowledgment

This work was supported by the French Grant "Action Intégrée MA/02/33".

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Éditeur
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http://www.inria.fr
ISSN 0249-6399